$\boxed{1}$ $y' = 5e^{(-2x)}(-2) = -10e^{-2x}$

$= -\dfrac{10}{e^{2x}}$

Entry Task: Find the derivatives of

1. $y = \dfrac{5}{e^{2x}}$ $= 5e^{-2x}$

2. $y = \sqrt{x^3 \ln(x^2+1)}$ $= (x^3 \ln(x^2+1))^{1/2}$

3. $R(x) = \dfrac{200x}{\ln(3x+3)}$

$\boxed{2}$ $y' = \frac{1}{2}(x^3 \ln(x^2+1))^{-1/2} \cdot \left(x^3 \frac{1}{x^2+1} \cdot (2x) + 3x^2 \ln(x^2+1)\right)$

$\boxed{3}$ $R'(x) = \dfrac{\ln(3x+3)200 - 200x \cdot \frac{1}{3x+3} \cdot 3}{(\ln(3x+3))^2}$

12.1/12.3 Antiderivatives and Integrals

Motivation: We have explored methods to go from a formula for a value to a formula for the rate of change of that value.

(*TR/TC to MR/MC, height to rate of ascent*, etc....)

But what if we know information about the rate first? Now we will explore how to go backward, from derivatives to "antiderivatives".

Example: Assume

$$f'(x) = 6x^5 - 4x^3 + 7$$

What is $f(x)$?

POWER SHOULD GO UP 1.

$f(x) = x^6 - x^4 + 7x$ ← IS ONE ANSWER

$f(x) = x^6 - x^4 + 7x + 14$ ← ANOTHER ANSWER

$f(x) = x^6 - x^4 + 7x - 17.2$

$f(x) = x^6 - x^4 + 7x + 1000017$

ALL SOL'NS LOOK LIKE

$$f(x) = x^6 - x^4 + 7x + \underbrace{\text{"A NUMBER"}}_{C \,=\, \text{"a constant"}}$$

We say $x^6 - x^4 + 7x + C$

is **the general antiderivative**.

The standard notation is **the (indefinite) integral** and we write

$$\int 6x^5 - 4x^3 + 7\,dx = x^6 - x^4 + 7x + C$$

Notes:

- $\int$ is called the *integral sign*.
- dx is called the *differential*, it tells us the variable in question and comes into play later when talking about definite integrals.
- The function we are integrating is called the *integrand*:

$$\int \text{"integrand"}\;dx$$

- We call "C" the *constant of integration* and it can be *any* constant.

In section 12.4, we will talk about how to find the constant, but we will need to be given additional information (*an initial condition*).

Example: Try to guess (and check) to find

$$\int 7x^6 + x^3 - \frac{1}{x} + e^x \, dx$$

$$x^7 + \quad x^4 \quad - \ln(x) + e^x + C$$

NEED SOMETHING HERE!

$$\frac{d}{dx} x^4 = 4x^3$$

NEED TO "CANCEL" the 4

$$x^7 + \frac{1}{4} x^4 - \ln(x) + e^x + C$$

SAME! YAY!

CHECK!

$$7x^6 + \frac{1}{4} 4x^4 - \frac{1}{x} + e^x$$

In this class, we are not going to learn any advanced integration methods (which dramatically simplifies things). If you are interested in more advanced methods take Math 125 (Calculus II). For basic integrals, here is what you do:

Step 0: Expand, simplify and rewrite powers.

Step 1: Identify the following:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad (\text{for } n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln(x) + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int k\,dx = kx + C$$

Step 2: Check by differentiating. Done!!

If it isn't one of the above, then you'll have to guess and check.

Example:

$$\int x - \frac{5}{\sqrt{x}} + e^{3x}\,dx$$

$$= \int x - 5x^{-1/2} + e^{3x}\,dx$$

$$= \frac{1}{2}x^2 - 5\frac{1}{1/2}x^{1/2} + \frac{1}{3}e^{3x} + C$$

$$= \boxed{\frac{1}{2}x^2 - 10x^{1/2} + \frac{1}{3}e^{3x} + C}$$

Check!

Example:

$$\int \frac{\sqrt[3]{t}}{4} + \frac{1}{6t^3} + 5t^{-1}\, dt$$

$$= \int \frac{1}{4} t^{1/3} + \frac{1}{6} t^{-3} + 5 \frac{1}{t}\, dt$$

$$= \frac{1}{4} \frac{1}{4/3} t^{4/3} + \frac{1}{6} \frac{1}{-2} t^{-2} + 5\ln(t) + C$$

$$= \boxed{\frac{3}{16} t^{4/3} - \frac{1}{12} \frac{1}{t^2} + 5\ln(t) + C}$$

CHECK ✓

Example:

$$\int \frac{2}{5x^{3/4}} + \frac{4}{7e^{5x}}\, dx$$

$$\int \frac{2}{5} x^{-3/4} + \frac{4}{7} e^{-5x}\, dx$$

$$= \frac{2}{5} \frac{1}{1/4} x^{1/4} + \frac{4}{7} \frac{1}{-5} e^{-5x} + C$$

$$= \boxed{\frac{8}{5} x^{1/4} - \frac{4}{35} e^{-5x} + C}$$

CHECK ✓

Example:

$$\int x^2(x^2-3)^2\,dx$$

$$(x^2-3)^2=(x^2-3)(x^2-3)$$

$$= x^4-6x^2+9$$

$$\int x^2(x^4-6x^2+9)\,dx$$

$$\int x^6-6x^4+9x^2\,dx$$

$$\frac{1}{7}x^7-\frac{6}{5}x^5+\frac{9}{3}x^3+C$$

$$\boxed{\frac{1}{7}x^7-\frac{6}{5}x^5+3x^3+C}$$

Check ✓

Example:

$$\int \frac{2x^4-3}{6x^5}\,dx$$

$$=\int \frac{1}{6x^5}(2x^4-3)\,dx$$

$$=\int \frac{2x^4}{6x^5}-\frac{3}{6x^5}\,dx$$

$$=\int \frac{1}{3}\frac{1}{x}-\frac{1}{2}x^{-5}\,dx$$

$$=\frac{1}{3}\ln(x)-\frac{1}{2}\frac{1}{-4}x^{-4}+C$$

$$=\boxed{\frac{1}{3}\ln(x)+\frac{1}{8}\frac{1}{x^4}+C}$$

Check ✓

Example: (from 12.4 HW)

Suppose $MR(q) = 14 - 0.2q$.

Find the formula for $TR(q)$

(Recall: $TR(0) = 0$, use this to find "C" at the end)

$$TR(q) = \int MR(q)\,dq = \int 14 - 0.2q\,dq$$

$$= 14q - 0.2\,\tfrac{1}{2}q^2 + C$$

$$TR(q) = 14q - 0.1q^2 + C$$

$$TR(0) = 0 \Rightarrow \quad 14(0) - 0.1(0)^2 + C = 0 \Rightarrow \boxed{C = 0}$$

$$\boxed{TR(q) = 14q - 0.1q^2}$$

Example: (from 12.4 HW)

Suppose $MC(q) = 30\sqrt{q+4}$.

And fixed costs are $FC = \$900$

Find the formula for $TC(q)$.

(Recall: $TC(0) = FC$, use this to find "C" at the end)

$$TC(q) = \int MC(q)\,dq$$

$$= \int 30\,(q+4)^{1/2}\,dq$$

$$= 30\,\frac{1}{3/2}(q+4)^{3/2} + C$$

$$= 30 \cdot \frac{2}{3}(q+4)^{3/2} + C$$

$$TC(q) = 20\,(q+4)^{3/2} + C$$

$$TC(0) = 900 \Rightarrow 20(0+4)^{3/2} + C \stackrel{?}{=} 900$$

$$20 \cdot 8 + C = 900$$

$$160 + C = 900$$

$$C = 740$$

Note: To complete this problem you had to guess and check a slightly varied version of the formula:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad \text{namely}$$

$$\int (x+a)^n\,dx = \frac{1}{n+1}(x+a)^{n+1} + C$$

You won't use this a lot, but are welcome to make a note of this more general version.

$$\boxed{TC(q) = 20(q+4)^{3/2} + 740}$$

Check